Accessed: 19/03/2010 04:21

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Source: *Isis*, Vol. 68, No. 3 (Sep., 1977), pp. 426-434
Published by: The University of Chicago Press on behalf of The History of Science Society
Stable URL: http://www.jstor.org/stable/231318

Accuracy of Planetary Theories, Particularly for Mars

By *Stanley E. Babb, Jr.**

THE CONTRAST BETWEEN the Ptolemaic and Copernican theories of planetary motion is a fascinating study. In order to gain some insights into the accuracy of representation of which these theories are basically capable, various orbits have been fitted to various versions of these theories using modern curve-fitting techniques and computational aids. Although this sort of approach does not illuminate the actual historical path which was trodden in the development or use of these theories, it does tend to give a basic geometric framework against which all of these theories can be viewed.

For the following discussion, all of the usual simplifications of the actual theories will be made. Thus consideration will be given to longitude only, by placing all deferents, epicycles, and so forth in the same plane, and the secular variations in any of the theories will be neglected.

The secular terms may be neglected for two reasons: one, planetary motion over a short interval of time is considered in this work—reaching a maximum of twenty-five years for a portion of an orbit of Saturn; two, in practice Ptolemy and Copernicus neglected the secular terms in the preparation of the planetary tables. These terms are usually a slow rotation of the apsidal line and produce only a second-order effect upon the planetary position.

Major attention is paid to Mars, for the deficiencies of the older theories become especially apparent when applied to Mars. Mercury has always presented special problems, and specialized models have been used. For these reasons Mercury will not be discussed.

DATA AND CALCULATIONS

"Observed" orbits were calculated from the *American Ephemeris and Nautical Almanac,*[1] from both a geostatic and a heliostatic standpoint. Such calculated orbits can be expected to diverge from actual observations[2] by less than 1".5, and thus the errors are well within those to be encountered in this paper. The orbit of Mars was also compared with the more precise treatment available,[3] with the same conclusion.

Received February 1975: revised/accepted August 1975.

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It is a pleasure to acknowledge the helpful discussions and critical reading of this manuscript by D. H. D. Roller and T. M. Smith.

[1] *The American Ephemeris and Nautical Almanac* (Washington, D.C.: U.S. Naval Observatory), vols. from 1948 through 1972.

[2] C. B. Watts and A. N. Adams, "Results of Observations Made with the Six Inch Transit Circle 1925–1941," *Publications of the U.S. Naval Observatory,* 1952, Ser. 2, *16*:367–445.

[3] R. L. Duncombe and G. M. Clemence, "Provisional Ephemeris of Mars 1950–2000," *U.S. Naval Observatory Circular No. 90,* Dec. 16, 1960.

These orbits, for Jupiter and Saturn, are different in one respect: the geostatic longitudes are corrected for minor perturbations, while the heliostatic longitudes are not. Since the perturbations are small,[4] they do not significantly change any of the orbital data for the considerations herein.

In addition, some artificial orbits were constructed: one from the Ptolemaic tables for planetary positions for Mars and one from a Keplerian ellipse with the appropriate eccentricity for Mars. This last orbit was calculated by a straightforward application of the area law of Kepler (see the appendix), which does not require the use of elliptic integrals, despite a statement to the contrary.[5]

Orbital parameters were determined by fitting the geometrical picture to the data, using a least-squares technique. This technique adjusted the parameters in the geometrical model until the sum of the squares of the error was minimized. Because of the complexity of the models, a grid search technique was used to locate the minima. This method is rather wasteful of computer time but is easily programmed.

RESULTS AND DISCUSSION

The results of the various calculations are most easily presented in tabular form, as seen in Tables 1 and 2. Not every possible combination of representation of orbits by models has been considered. Not all are of sufficient interest to justify the computer time involved. The results presented in the tables are for $\sqrt{\Sigma}$(calculated position − observed position)^2/(number of position − 1), that is, the root-mean-square (rms) positions are the positions defined by the particular orbit under consideration and the calculated positions are those given by the geometrical model for the values of the parameters which minimizes this deviation. Each line in Table 1 and column in Table 2 represents a separate calculation, independent of all others. A more detailed presentation of two results is also presented in Figure 1.

The accuracy of observations available to Ptolemy and Copernicus is generally taken to be about 10′. This figure is based very loosely upon direct statements[6] and can give rise to the feeling that the theories themselves were accurate to 10′. This is possibly true for Jupiter, Saturn, and Venus; it is not true for Mars. For Mars the combination (in Ptolemy) of a large epicycle radius with a large eccentricity serves to make it impossible for the theory to be better than 10′ of arc. There is insufficient distinction between the geometrical models of Copernicus and Ptolemy to make the former better—in fact it is worse if the same number of adjustable parameters are used.

Planetary theory before Ptolemy lacked the famous equant which he introduced. The first line in Table I represents the results using the standard Ptolemaic theory of deferent, epicycle, eccentric, and equant. The third line represents a theory of only deferent epicycle and eccentric; that is, the equant has been omitted and the parame-

[4]G. M. Clemence, "The Perturbations of the Five Outer Planets by the Four Inner Ones," Astronomical Papers prepared for the use of *The American Ephemeris and Nautical Almanac*, Vol. XIII, Part V (Washington, D. C.: U.S. Naval Observatory, 1954).

[5]D. J. Price, "Contra Copernicus," in *Critical Problems in the History of Science*, ed. M. Clagett (Madison: University of Wisconsin Press, 1959), pp. 197–218.

[6]E.g., Kepler stated: "Ptolemaeus vero profitetur, se infra X minuta seu sextam partem gradus observando non descendere" (J. Kepler, *Astronomia nova Αιτιολογητος, sev physica coelestis tradita commentariis de motibvs stellae Martis, ex observationibus G. V. Tychonis Brahe,* Pragae, 1609, p. 113), and Rheticus quoted Copernicus saying he would be delighted if he could make his planetary theory agree with observed positions within 10′ (quoted by J. L. E. Dryer, *A History of Astronomy from Thales to Kepler,* 2nd ed., New York: Dover, 1953, pp. 343–344). In the following this 10′ figure will be loosely adopted for discussion purposes.

Table 1. RMS errors, in minutes, of various representations for Mars

Model	Orbit					
	Calculated from ephemeris	Calculated from *Almagest*	Calculated for ellipse	d_1	d_2	R_2
				for ephemeris data		
Geostatic longitudes						
Ptolemy:						
standard	12.0	1.97		0.09444	= 0.09444	0.65229
modified	9.9	1.93		0.11297	0.08588	0.66111
without equant	26.2				0.13694	0.60573
tabular	12.7			0.09393	0.09401	0.65216
Al-Shāṭir/ Copernicus	11.0			0.05841	0.14085	0.66411
Heliostatic longitudes						
Ptolemy:						
standard deferent	5.96		5.40	0.0936	= 0.0936	
modified deferent	1.01		0.57	0.07069	0.11652	
Al-Shāṭir/ Copernicus	1.11		0.79	0.03361	0.15390	
Kepler:						
ellipse	0.88			$e = 0.09336$		

Ptolemaic: d_1 = equant–center distance
d_2 = eccentric–center distance
R_2 = epicycle radius

Al-Shāṭir: d_1 = first epicycle radius
d_2 = eccentric–center distance
R_2 = second epicycle radius (earth's orbital radius for Copernicus)

ters recomputed. It is seen that the theory which lacks the equant is only capable of representing this particular orbit with an accuracy of 26′.2 rms, while the introduction of the equant will lower this figure to 12′.0. Thus Ptolemy's introduction of the equant served to more than double the accuracy of which the theory is capable. The practical advantages of the introduction of the equant are clear.

One of the assumptions made by Ptolemy is that the eccentric-center distance equals the center-equant distance. This rather specialized assumption can be easily relaxed, and if one does so the theory is improved slightly—it is now possible to represent the planetary positions to within 9′.9. Considerable additional labor is required for the determination of the parameters of this particular model.

Adopting the standard identification that in the heliocentric theory the Ptolemaic deferent represents the orbit of the planet around the sun and the Ptolemaic epicycle represents the effects of the earth's orbit (for superior planets), one may legitimately ask how much of the observed errors are due to each identification. This has been done before,[7] apparently incorrectly. The appropriate question is how well the Ptolemaic deferents can fit the heliostatic orbit of Mars. Here the angles are heliocentric longitudes—that is, those as seen from the sun, not from the earth.

[7] Price, "Contra Copernicus."

The results are rather surprising: the standard Ptolemaic deferent represents the observed heliostatic orbit of Mars to within slightly less than 6′. This 6′ can be compared to the results of representing a Keplerian ellipse of eccentricity appropriate to Mars by the Ptolemaic deferent, which are of the same order, 5′.4. If now the equality of the center-eccentricity and center-equant distances is relaxed, the results become almost astonishing: this modified Ptolemaic deferent fits the observed orbit to within 1′ and fits the orbit calculated from a Keplerian ellipse to within 0′.7. This is precisely the curve which led to the famous seventy trials of Kepler[8] in the successive approximations for the determinations of the parameters. This curve represented the data well, but other considerations, such as latitudes and the eccentricity calculated from the solar position, are what led Kepler to the renowned 8′ discrepancy which he said started him in the considerations toward his later developments. This result using longitudinal angles alone also implies why distances played such an important role in later developments.

There is another aspect of Ptolemaic and Copernican theory which has been inadequately stressed and on which most authors are silent. Although the geometrical models of the planetary motion are usually discussed, in practice these were not used for the calculation of planetary positions. The *Almagest* contains a reduction of the elegant theory to numerical tables which can be and were used for the calculation of planetary positions. The scheme for the use of these tables was given by Ptolemy and was used with but moderate modification by later authors until Kepler changed the entire scheme.

The problem lies not in the angles concerned with the deferent, but with the calculation of the angle subtended at the observer by the epicycle radius. This angle is a function of the angle between the apsidal line and the line from the equant to the epicycle center (longitude), as well as that between the observer-epicycle and the epicycle radius (anomaly). In order to reduce this complex function to easy computational tabulation, Ptolemy used an approximation, reducing the calculation to a multiplication and an addition. Thus he first calculated this angle as seen by the observer as if the observer-to-center-of-the-epicycle distance is the standard distance. He then calculated this angle as if the distance were the maximum, and finally as if it were the minimum. The tabulated quantities are the angles for the standard distance and the differences between this angle and that calculated for minimum distance. These are the functions which Neugebauer[9] calls c_5, c_6, and c_7. Since now c_5 and c_7 represent corrections which would not be fully applied for intermediate distances, an interpolation function c_8 is also tabulated, and either c_5 or c_7 is multiplied by this function before being added into the final answer for the planet's position. The function c_8 is calculated by assuming that the correction would be in the same ratio as the maximum angle subtended by the epicycle's radius. Then the correction factor was calculated from: $c_8 = (\arcsin(R_2/X_2) - \arcsin(R_2/R_1))/(\arcsin(R_2/(R_1 + e)) - \arcsin(R_2/R_1))$, where R_2 is the epicycle radius, R_1 that of the deferent, e the eccentricity, and X_2 is the distance from the observer to the center of the epicycle, and

[8]Kepler, *Astronomia nova,* p. 95. Kepler's 70 iterations represent several determinations; see Owen Gingerich, "Kepler's Treatment of Redundant Observations," *Internationales Kepler-Symposium, Weil der Stadt, 1971,* ed. F. Krafft, K. Meyer, and B. Sticker (Hildesheim: Gerstenberg, 1973), p. 307. It is also of interest to compare these results with Kepler's: he obtained 0.11332 and 0.07232, which should be compared with 0.11652 and 0.07069 from Table 1. Despite the differences in data sets and criterion of fit the agreement is remarkable. The rms deviation calculated from Kepler's figures is 1′.2, which should be compared with the 1′ from Table 1.

[9]O. Neugebauer, *The Exact Sciences in Antiquity* (New York: Dover, 1969), pp. 201 ff.

here $X_2 > R_1$; the expression $R_1 + e$ is replaced by $R_1 - e$ for the case $X_2 < R_1$.

For Venus, Jupiter, and Saturn the corrections so introduced are small enough that the inaccuracies of the operation are negligible. For Mars, however, the combination of the large epicycle and large eccentricities leads to difficulties: the errors introduced can range from + 4.′7 to - 13.′1. Hence the errors are of the same order as the basic inaccuracies of the theory itself.

The exact error introduced at any particular position depends upon both longitude and anomaly, and thus its effect upon the results is variable. In Figure 1 the errors of the standard Ptolemaic theory are presented; the dashed lines represent the extreme effects of this "tabular inaccuracy" upon the possible calculated positions. Depending upon the exact anomaly for which the calculation is made, the error at any particular longitude could vary between the dashed lines. Thus the theory could vary with errors between +19 and -25.′

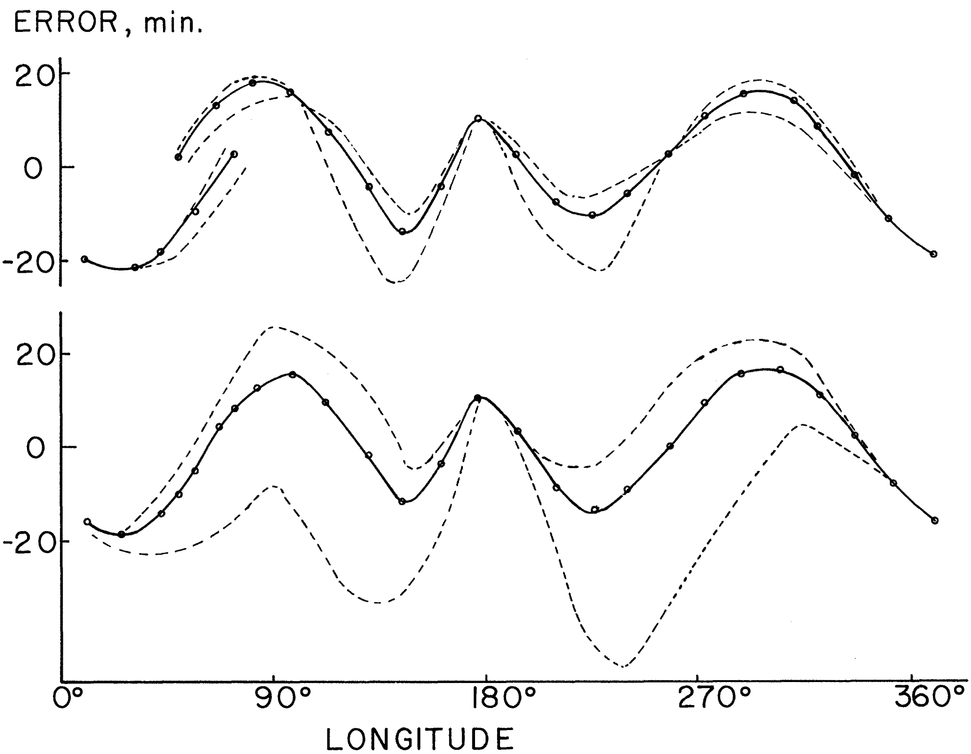

Figure 1. The errors of fit of (top) the standard Ptolemaic theory and (bottom) the Copernican theory to the same data for Mars. The dotted lines represent the possible additional errors from the tabular errors.

Because the tabular orbit does not correspond exactly to the geometric orbit, an orbit was calculated from the tables and considered with both the standard Ptolemaic theory and the modified Ptolemaic theory; the resulting errors are approximately 2.′ Also it would be possible to calculate the orbital parameters using only the tabular approach, though the calculations are considerably more involved. Thus the tabular theory based upon the geometry with center-equant distance not equal to center-eccentric distance is only capable of representing the observed orbit to within rms errors of 12.′7—almost a one-third increase in error. The constants for this tabular approach are also slightly different.

The Alfonsine tables were of central importance in European astronomical calculations of planetary positions until replaced by Keplerian astronomy. A comparison of these Alfonsine tables with those in the *Almagest* indicates remarkable agreement—except in the tabulated interpolation coefficient c_8. At first sight it would appear to have been calculated differently, but this is only because the column is tabulated in terms of the longitude of the epicycle center as seen from the earth, rather than as seen from the equant. The values of the parameters from which the tables were calculated appear to be identical to the Ptolemaic values, and not to the differing ones which have been cited.[10] (This same source cites an interpolation scheme which bears no discernible resemblance to that used in any tables known to me.) For Jupiter there is a contradiction in that the tables are internally inconsistent. The value of the eccentricity used for calculating the $c_3 + c_4$ column is apparently 3;7 and not that used for c_5 and c_7 columns, which were calculated with the standard Ptolemaic value of 2;45.

The second major system is the Copernican one, or that of Ibn al-Shāṭir,[11] who preceded him. Their planetary constructions are geometrically identical. Here the equant is replaced by a small epicycle circling in a sense opposite to that of the deferent motion. This construction results in a path which is wider, rather than narrower, at the 90° points, a point which Kepler notes.[12] Thus the al-Shāṭir representation of the planetary position (or the standard Copernican representation of the deferent) is inherently inferior to that of Ptolemy. The difference in the angle subtended by the rotating point on the Copernican epicycle from that in the standard Ptolemaic deferent is small—as Neugebauer[13] has remarked (but for Mars, it is not within Neugebauer's 1′)—and tends to increase rather than decrease the error. Thus one could anticipate that the al-Shatir/Copernicus representation of planetary positions would be inferior to that of a modified Ptolemaic theory, and such is indeed the case, as can be seen from Table I.

In making the comparison between the Ptolemaic and Copernican representations of given orbits it is well to remember that Ptolemy effectively determined two distances from the data—the epicycle radius and the eccentricity. Copernicus effectively determined three—the eccentricity, the epicycle radius, and the radius of the earth's orbit in relationship to those two. Thus the Copernican theory has an additional parameter and so should be compared with the modified Ptolemaic theory rather than with the standard; when this is done, both theories are found to have the same number of adjustable constants. Neugebauer's contention[14] that Copernicus simply took one-third and two-thirds of the Ptolemaic constants—that is, did not have three adjustable constants, only two—is rendered doubtful by the parameters chosen for Mars and Venus, for they violate this rule. The constraints imposed by the

[10] D. Price, *The Equatorie of the Planetis* (Cambridge: Cambridge University Press, 1955), pp. 114 ff. The erroneous values would appear to be a misquote of M. Delambre, *Histoire de l'astronomie du moyen âge* (Paris: V^e Courcier, 1819), p. 254. The same author also apparently takes values for maximum angles as the standard Ptolemaic eccentricity and epicycle radius, thus perhaps misquoting E. S. Kennedy, "A Survey of Islamic Astronomical Tables," *Transactions of the American Philosophical Society*, 1956, *46*:172, for al-Khwārizmī and probably also for al-Battānī.

[11] E. S. Kennedy and V. Roberts, "The Planetary Theory of Ibn al-Shātir," *Isis*, 1959, *50*:227–235.

[12] Kepler, *Astronomia nova*, p. 14.

[13] O. Neugebauer, "On the Planetary Theory of Copernicus," *Vistas in Astronomy*, 1968, *10*:89–103.

[14] *Ibid.*

data are as great as or greater than the desire to agree with the Ptolemaic deferent at only two points.

The tabular inaccuracies are much worse for Copernicus than for Ptolemy. Instead of using one function for $X_2<R_1$ and another for $X_2>R_1$, a single interpolation function of the Ptolemaic type is used to cover the entire range. This means the maximum addition possible is large (13° compared with 8°), so errors are magnified and can rise to +13, -34′ for the Copernican figures for Mars.

The lower curve in Figure 1 is the fit to the same data for Mars using the al-Shāṭir scheme as the upper curve uses the Ptolemaic, again with the dashed lines indicating the effects of tabular errors. It is not clear why this curve is closed and the Ptolemaic one is open: one would not expect the curve to close, since the differences in anomaly would change the errors at a given longitude. The tabular inaccuracies coupled with the errors of the geometric model can give rise to total errors of +27, -45′.

Finally a Keplerian ellipse was fitted to the same heliostatic data. Here the errors are entirely due to perturbations and are correspondingly small, less than 1′, despite the fact that fewer parameters are being determined in this new and better geometrical representation.

Turning briefly to the other planets, results for Venus, Jupiter, and Saturn are contained in Table 2, along with the maximum errors and the tabular errors for the Copernican scheme of interpolation. The tabular errors for Ptolemy are less than those for Copernicus. In all cases the rms deviation (but not necessarily the maximum deviation) of the theory from the observations is well within the presumed 10′ figure, and the tabular errors are also negligible.

The results presented herein are somewhat data dependent. Using a differing data set could result in slightly different constants and thus slightly different error figures. Because of the structure of both the theory and the data, however, these variances should be relatively small. To illustrate this, the standard Ptolemaic theory was fitted with various subsets of the entire orbit—5 points were taken rather than the 25 usually used. This was done for each set of 5 consecutive points, as well as 5 distributed over the whole orbit. If one considers only the "best" of the subset fits (rms deviations of 15′ over the whole set, compared with 12 if the whole set is used) a variation of 11 per cent for the eccentricity and 0.8 per cent for the epicycle radius is encountered. Since the fit is poorest for Mars, lesser variations would be encountered for other planets.

As a further illustration, it would be possible to use a different criterion for determining the parameters to be used. For example, it would be closer to the historical practice to attempt to minimize the maximum deviation of the observed position from that calculated from the model rather than from the least-squares procedure. This different criterion results in surprisingly little difference: for the modified Ptolemaic theory in the geostatic model for Mars the maximum difference in the parameters and those given in Table 1 is only 4 per cent. The maximum deviation drops from 16′.9 to 15′.9 and the rms deviation increases from 9′.9 to 10′.9. Thus it may be concluded that these results are relatively insensitive to the criterion used for parameter determination.

When comparing positions of the planets computed from the tables which resulted from these theories with positions retrodicted from modern theories, the errors encountered will include those of a basic theory's faults, plus those which arise from poor parameter selection, plus the problems with secular terms which become important when converting time to angles over long periods. Thus Gingerich's

Table 2. RMS deviations, in minutes, for Venus, Jupiter, and Saturn

	Geocentric Orbits			
	Standard Ptolemy	Modified Ptolemy	Al-Shātir/ Copernicus	Tabular error[a] maximum/minimum
Venus	6.12	6.06	6.09	0′.4–1′.2
d_1	0.01439	0.01506	0.007509	
d_2	0.01439	0.01383	0.02143	
R_2	0.7218	0.7216	0.7217	
max error	11	11	11	
Jupiter	7.05	6.59	6.66	7″.5–9″.8
d_1	0.06254	0.05118	0.02489	
d_2	0.06254	0.1282	0.09133	
R_2	0.1498	0.1357	0.1386	
max error	13	12	14	
Saturn	2.16	1.96	1.95	3″.8–4″.4
d_1	0.05012	0.05000	0.02573	
d_2	0.05012	0.01542	0.03986	
R_2	0.1088	0.1437	0.1438	
max error	5.8	4.4	4.4	

[a]Computed for the parameters used by Copernicus in preparing his tables, not the parameters developed herein.

statement[15] that the Prutenic tables can be off by up to 2° for Mars would indicate that roughly 45′ (the maximum possible deviation shown in Fig. 1) are from the poor theoretical representation plus tabular errors and 75′ are from poor parameter selection and secular term errors.

In comparing the results of this work with the only previous such estimation known to me,[16] the results are not in very good agreement. The maximum errors estimated by Price (which do not include the tabular errors) are too small for Venus and Jupiter and too large for Mars. He does not give his computations; consequently it is not possible to see how he reached the published values.

CONCLUSIONS

The standard Ptolemaic theory and the Copernican theory of planetary motions are capable of reproducing the motions of the planets—if the parameters are chosen properly—to well within the observational uncertainties available for Saturn, Jupiter, and Venus before Tycho Brahe. For Mars, however, the predictions of the theories are capable of given rms deviations only comparable to observational uncertainty. The poor reproduction in tabular form of the geometrical concepts of the theory indicates that deviations for Mars may have considerably exceeded observational uncertainty, particularly for the Copernican theory. These "tabular errors"—not to be confused with computational mistakes in calculating the tables—are not well known and are only significant for Mars.

Finally it should be remembered that the famous Alfonsine tables are not based upon a redetermination of the planetary astronomical constants of the Ptolemaic

[15]O. Gingerich, "On Examination of the Rudolphine Tables," *XIth International Congress of the History of Science,* Vol. II, Pt. 3, p. 31.

[16]Price, "Contra Copernicus."

theory, for the Alfonsine tables used the same constants found in the *Almagest,* except for an inconsistency in the case of Jupiter.

APPENDIX

For computational purposes the easiest representation of the models is to calculate the x and y positions of the planet from the geometrical model, use these to construct the tangent of the observed angle, subtract the longitude from the aspidal line as seen from the equant, and simplify. The results for the Ptolemaic theory are

$$\theta_{obs} = \theta_1 + \arctan \frac{R_2 \sin \theta_4 + (d_1 + d_2) \cos \theta_1}{1 - d_1^2 \cos^2\theta_1 + R_1 \cos \theta_4 + d_2 \sin \theta_1}$$

where θ_1 was measured from an initial position 90° to the aspidal line for no significant reason (R_1, d_1, etc. are defined in Table I).

For the Shāṭir/Copernicus construction a similar approach yields

$$\theta_{obs} = \theta_1 + \arctan \frac{R_2 \sin \theta_4 - (d_1 + d_2) \sin \theta_4}{1 + R_2 \cos \theta_4 + (d_2 - d_1) \cos \theta_1}$$

where θ_1 is now measured from the aspidal line and for the heliostatic orbit R_2 is set equal to zero. For all of these formulas the radius of the deferent is taken as unity.

For the Keplerian ellipse one proceeds from the areal law $dA/dt = \text{constant} = \frac{1}{2} r^2 d\theta/dt]$ Using the polar form of the ellipse $r = l/(1 + e \cos \theta)$, the resulting integral can be evaluated to give

$$\frac{1}{1 - e^2} \left\{ \frac{2}{(1 - e^2)^{1/2}} \arctan \left[\left(\frac{1 - e}{1 + e} \right)^{1/2} \tan \frac{\theta}{2} \right] - \frac{e \sin \theta}{1 + e \cos \theta} \right\} = \text{const}\,(t - t_0)$$

which gives the time as an explicit function of angle. The constant is evaluated by requiring that the orbit close with the correct period.

For least-squares analysis the functions were evaluated, and the sum of $(\theta_{obs} - \theta_{calc})^2$ was minimized. For the ellipse it was more convenient to minimize the time differences and to convert the rms time error to rms angle error by multiplying by the mean motion—accurate enough for present purposes.